

# NAHM'S CONJECTURE AND Y-SYSTEM

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**ABSTRACT.** Nahm's conjecture relates  $q$ -hypergeometric modular functions to torsion elements in the Bloch group. An interesting class of such functions can be (conjecturally) obtained from a pair of Dynkin diagrams of  $ADET$  type. Using properties of Y-system, we prove that for a matrix of the form  $A = \mathcal{C}(X) \otimes \mathcal{C}(X')^{-1}$  where  $\mathcal{C}(X)$  and  $\mathcal{C}(X')$  are one of  $ADET$  Cartan matrices, every solution of the equation  $\mathbf{x} = (1 - \mathbf{x})^A$  gives rise to a torsion element of the Bloch group.

## 1. INTRODUCTION

In [N], Nahm considered a question of when a  $q$ -hypergeometric series is modular and made a conjecture relating this question to algebraic K-theory, motivated by integrable perturbations of rational conformal field theories.

**Definition 1.1.** Let  $A$  be a positive definite symmetric  $r \times r$  matrix,  $B$  be a vector of length  $r$ , and  $C$  a scalar, all three with rational entries.

We consider a  $q$ -hypergeometric series

$$f_{A,B,C}(z) = \sum_{n=(n_1, \dots, n_r) \in (\mathbb{Z}_{\geq 0})^r} \frac{q^{\frac{1}{2}n^t A n + B^t n + C}}{(q)_{n_1} \dots (q)_{n_r}}$$

where  $q = e^{2\pi i z}$  and  $(q)_n = (1-q)(1-q^2)\dots(1-q^n)$  denotes the Pochhammer symbol. If  $f_{A,B,C}$  is a modular function, then we call  $(A, B, C)$  a modular triple and  $A$  the matrix part of it.

The most famous examples are the Rogers-Ramanujan identities:

$$\sum_{n=0}^{\infty} \frac{q^{n^2-1/60}}{(q; q)_n} = \frac{q^{-1/60}}{(q; q^5)_{\infty} (q^4; q^5)_{\infty}}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n+11/60}}{(q; q)_n} = \frac{q^{11/60}}{(q^2; q^5)_{\infty} (q^3; q^5)_{\infty}}.$$

In this case, we have modular triples  $((2), (0), -1/60)$  and  $((2), (1), 11/60)$ .

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*Date:* Sep 15, 2011.

As an attempt to characterize matrix parts of modular triples, one considers the asymptotic behavior of  $f_{A,B,C}$  when  $z$  approaches 0 and is led to a system of equations associated to the matrix  $A = (a_{ij})$  given by

$$(1.1) \quad x_i = \prod_{j=1}^r (1 - x_j)^{a_{ij}}, \quad (i = 1, \dots, r)$$

This is a system of  $r$  equations of  $r$  variables  $x_1, \dots, x_r$ . When there is no confusion, I will denote this system of equations by  $\mathbf{x} = (1 - \mathbf{x})^A$ .

For a solution  $\mathbf{x} = (x_1, \dots, x_r)$  of  $\mathbf{x} = (1 - \mathbf{x})^A$ , we consider a formal sum  $\xi_{\mathbf{x}} = [x_1] + \dots + [x_r]$  in the group ring  $\mathbb{Z}[F]$  of a certain number field  $F$  depending on the solution  $(x_1, \dots, x_r)$ . This can be regarded as an element of the Bloch group  $\mathcal{B}(F)$ , whose definition is given in the section 2.

Nahm's conjecture is as follows :

**Conjecture 1.2.** Let  $A$  be a positive definite symmetric  $r \times r$  matrix with rational entries. The following are equivalent:

- (i) For any solution  $x = (x_1, \dots, x_r) \in F^r$  of 1.1, the element  $\xi_x \in \mathcal{B}(F)$  is a torsion element of  $\mathcal{B}(F)$
- (ii) There exists a modular triple  $(A, B, C)$ .

This conjecture is discussed in detail in Nahm's paper [N] from the viewpoint of conformal field theory and in Zagier's [Zag1] from that of number theory. Recently, Zwegers and Vlasenko found counterexamples to the above conjecture in [VZ]. Thus the conjecture must be modified and reformulated although this issue is not touched here.

It has been long known (and conjectured) that there is an interesting class of modular triples whose matrix part is given by the Kronecker product of a pair of Cartan matrices of *ADET* types. To be precise, let us consider a matrix  $A = \mathcal{C}(X) \otimes \mathcal{C}(X')^{-1}$  where  $X$  and  $X'$  are one of *ADET* Dynkin diagrams and  $\mathcal{C}(X)$  and  $\mathcal{C}(X')$  their Cartan matrices. See the section 3.2 for definitions.

For such matrices  $A$ , many modular triples have been found. In the case of Rogers-Ramanujan identity, the matrix part of the modular triple is  $(2) = \mathcal{C}(A_1) \otimes \mathcal{C}(T_1)^{-1}$ . The Andrews-Gordon identity is a well-known generalization of Rogers-Ramanujan identity and it provides a modular triple with its matrix part  $A = \mathcal{C}(A_1) \otimes \mathcal{C}(T_n)^{-1}$ .

In this paper, I provide a proof of the following theorem.

**Theorem 1.3.** Let  $A = \mathcal{C}(X) \otimes \mathcal{C}(X')^{-1}$  where  $(X, X')$  is a pair of *ADET* Dynkin diagrams. For every solution  $\mathbf{x} = (x_i)_{i \in \mathbf{I}}$  of the equation  $\mathbf{x} = (1 - \mathbf{x})^A$  in a number field  $F$ ,  $\xi_{\mathbf{x}} = \sum_{i \in \mathbf{I}} [x_i]$  is a torsion element of the Bloch group  $\mathcal{B}(F)$ .

The proof is obtained using properties of Y-system whose definition is given in the section 3. Frenkel and Szenes studied dilogarithm identities and their relation to torsion in algebraic K-theory [FS1] and Y-system [FS2]. In [N, section 4], Nahm briefly explains how one can obtain a proof of the above statement assuming the periodicity of Y-system. It seems that, however, more structural properties of Y-system need to be used to complete the proof. Nakanishi's paper [Nak] contains most of results used here except relating results to the Bloch group.

Y-system, which can be defined for a pair of Dynkin diagrams, turns out to be very useful to study the equation  $\mathbf{x} = (1 - \mathbf{x})^A$ . We can relate this equation to a Y-system and then using properties of it, we can show that all solutions give torsion elements of the Bloch group.

Many conjectured properties of Y-system, originated from thermodynamic Bethe ansatz approach of conformal field theory [Zam], had remained open for years but now have been proved rigorously due to recent development of theory of cluster algebras. (See [FZ] and [K]). One may hope that the correct reformulation of the Nahm's conjecture incorporates this and it would help us find new directions toward understanding modular triples and modular  $q$ -hypergeometric series.

In sections 2 and 3, I give necessary definitions and properties of the Bloch group and Y-system. The proof of theorem 1.3 is given in the section 4.

**Acknowledgements.** I wish to thank An Huang and Richard Borcherds for helpful discussions and Nicolai Reshetikhin for suggesting to me to learn cluster algebras and Edward Frenkel for explaining me his works.

## 2. BLOCH GROUP

In this section, I define the Bloch group for a field and the Bloch-Wigner dilogarithm function. See [Zag1] and [Zag2] for a more thorough treatment.

**Definition 2.1.** Let  $F$  be a field.  $\Lambda^2 F^*$  denotes abelian group of formal sums of  $x \wedge y, x, y \in F^*$  modulo the relations  $x \wedge x = 0, (x_1 x_2) \wedge y = x_1 \wedge y + x_2 \wedge y$  and  $x \wedge (y_1 y_2) = x \wedge y_1 + x \wedge y_2$ .

Let  $\partial : \mathbb{Z}[F^* \setminus \{1\}] \rightarrow \Lambda^2(F^*) \otimes_{\mathbb{Z}} \mathbb{Q}$  be a  $\mathbb{Z}$ -linear map defined by  $\partial([x]) = x \wedge (1 - x)$ . Let  $A(F) = \ker \partial$  and  $C(F)$  the subgroup of  $A(F)$  generated by the elements

$$(2.1) \quad [x] + [1 - xy] + [y] + \left[\frac{1 - y}{1 - xy}\right] + \left[\frac{1 - x}{1 - xy}\right]$$

$$(2.2) \quad [x] + [1 - x]$$

$$(2.3) \quad [x] + \left[\frac{1}{x}\right].$$

It is convenient to set  $[0] = [1] = [\infty] = 0$  in  $A(F)$ . We call 2.1 the five-term relation. The Bloch group  $\mathcal{B}(F)$  of  $F$  is defined by  $\mathcal{B}(F) = A(F)/C(F)$ .

The dilogarithm function is defined by

$$\text{Li}_2(z) = - \int_0^z \frac{\ln(1-t)}{t} dt, z \in \mathbb{C} - [1, \infty).$$

The Bloch-Wigner dilogarithm function is a useful tool to study the Bloch group and is given by

$$D(z) = \text{Im}(\text{Li}_2(z)) + \log |z| \arg(1-z).$$

$D(z)$  is a real analytic function on  $\mathbb{C}$  except at 0 and 1, where it is continuous but not differentiable. Since  $D(\bar{z}) = -D(z)$ , it vanishes on  $\mathbb{R}$ . It satisfies following functional equations

$$(2.4) \quad D(x) + D(1-xy) + D(y) + D\left(\frac{1-y}{1-xy}\right) + D\left(\frac{1-x}{1-xy}\right) = 0$$

$$(2.5) \quad D(x) + D(1-x) = D(x) + D\left(\frac{1}{x}\right) = 0.$$

$D(z)$  can be used to define a map from  $B(\mathbb{C})$  to  $\mathbb{R}$ . For  $\xi = \sum_i n_i [x_i] \in B(\mathbb{C})$ , let  $D(\xi) = \sum_i n_i D(x_i)$ . By 2.4 and 2.5, it is well-defined. For a number field  $F$ , an embedding  $\sigma : F \hookrightarrow \mathbb{C}$  and  $\xi \in \mathcal{B}(F)$ , we may consider  $D(\sigma(\xi))$ . If  $D(\sigma(\xi)) = 0$  for all such embeddings  $\sigma$ , then  $\xi \in \mathcal{B}(F)$  is a torsion element in  $\mathcal{B}(F)$ . [Zag2, section 4]

Let  $\mathbb{C}(y)$  be a field of rational functions in  $(y_i)_{i \in I}$  where  $I$  is a finite set. For  $f \in \mathbb{C}(y)$ ,  $f|_{\mathbf{a}}$  means the evaluation of  $f$  at  $(y_i) = \mathbf{a} = (a_i) \in \mathbb{C}^n$ .

**Proposition 2.2.** *Suppose that we have a set  $S$  of rational functions  $f_i \in \mathbb{C}(y)$  such that  $\sum_{f_i \in S} (1 - f_i) \wedge f_i = 0$  in  $\Lambda^2 \mathbb{C}(y)^*$ .  $\sum_{f_i \in S} D(f_i|_{\mathbf{x}})$  is independent of  $\mathbf{x} \in \mathbb{C}^n$ .*

See [Zag1, chapter II. section 2.A.] and references therein.

To obtain such a set of rational functions satisfying the condition of the above statement, we now turn our attention to Y-system. Y-system is a good supplier for such rational functions!

### 3. Y-SYSTEM

**3.1. Y-system for a pair of ADE Dynkin diagrams.** In this section, I closely follow the notations of [Nak].

Let  $X$  be a Dynkin diagram of ADE type with the index set  $I$ . The rank and the Coxeter number of  $X$  will be denoted by  $r$  and  $h$ . We write Cartan matrix of  $X$  as  $\mathcal{C}(X)$  and the adjacency matrix  $\mathcal{I}(X) = 2 \text{id} - \mathcal{C}(X)$ .

We give an alternate bicoloring on the pair of Dynkin diagrams. To be precise, we call a decomposition  $I = I_+ \cup I_-$  bipartite if  $\mathcal{I}(X)_{ij} = 1$  implies  $(i, j) \in I_+ \times I_-$  or  $(i, j) \in I_- \times I_+$ .

Now consider an ordered pair of Dynkin diagrams  $(X, X')$ . For another Dynkin diagram  $X'$ ,  $I' = I'_+ \cup I'_-$ ,  $r'$ ,  $h'$ ,  $\mathcal{C}(X')$ , and  $\mathcal{I}(X')$  will be defined analogously. Let us fix bipartite decompositions  $I$  and  $I'$ . Let  $\mathbf{I} = I \times I'$  and  $\mathbf{I} = \mathbf{I}_+ \sqcup \mathbf{I}_-$  where  $\mathbf{I}_+ = (I_+ \times I'_+) \sqcup (I_- \times I'_-)$  and  $\mathbf{I}_- = (I_+ \times I'_-) \sqcup (I_- \times I'_+)$ .

Let  $\epsilon : \mathbf{I} \rightarrow \{1, -1\}$  be the function defined by  $\epsilon(\mathbf{i}) = \pm 1$  for  $\mathbf{i} \in \mathbf{I}_\pm$  and  $P_\pm = \{(\mathbf{i}, u) \in \mathbf{I} \times \mathbb{Z} | \epsilon(\mathbf{i})(-1)^u = \pm 1\}$ . Roughly speaking, we want our alternate bicoloring interchanges their colors as  $u \in \mathbb{Z}$  changes by 1.

**Definition 3.1.** For a family of variables,  $\{Y_{ii'}(u) | i \in I, i' \in I, u \in \mathbb{Z}\}$ , the Y-system  $\mathbb{Y}(X, X')$  associated with a pair  $(X, X')$  of *ADE* Dynkin diagram is defined as a system of recurrence relations as follows :

$$(3.1) \quad Y_{ii'}(u-1)Y_{ii'}(u+1) = \frac{\prod_{j:j \sim i} (1 + Y_{ji'}(u))}{\prod_{j':j' \sim i'} (1 + Y_{ij'}(u)^{-1})}$$

where  $a \sim b$  means  $a$  is adjacent to  $b$ . Note that Y-system  $\mathbb{Y}(X, X')$  consists of two decoupled copies,  $\{Y_{\mathbf{i}}(u) | (\mathbf{i}, u) \in P_+\}$  and  $\{Y_{\mathbf{i}}(u) | (\mathbf{i}, u) \in P_-\}$ . If  $(\mathbf{i}, u) \in P_+$ ,  $Y_{\mathbf{i}}(u)$  can be written as a rational function of variables  $\{Y_{\mathbf{i}}(0) | \mathbf{i} \in \mathbf{I}_+\}$  and  $\{Y_{\mathbf{i}}(1) | \mathbf{i} \in \mathbf{I}_-\}$  whereas if  $(\mathbf{i}, u) \in P_-$ ,  $Y_{\mathbf{i}}(u)$  only depends on  $\{Y_{\mathbf{i}}(0) | \mathbf{i} \in \mathbf{I}_-\}$  and  $\{Y_{\mathbf{i}}(1) | \mathbf{i} \in \mathbf{I}_+\}$ .

If a solution  $\{Y_{\mathbf{i}}(u)\}$  of Y-system does not have any dependence on  $u$  so that  $Y_{\mathbf{i}}(u) = y_{\mathbf{i}}$  for each  $\mathbf{i}$  in a field, we call it a constant Y-system. Note that this is a system of  $rr'$  equations of  $rr'$  variables

$$(3.2) \quad y_{ii'}^2 = \frac{\prod_{j:j \sim i} (1 + y_{ji'})}{\prod_{j':j' \sim i'} (1 + y_{ij'}^{-1})}.$$

Now I state several results about Y-system relevant to our proof of theorem 1.3. For all these theorems below, we assume that  $\{Y_{\mathbf{i}}(u) | \mathbf{i} \in \mathbf{I}, u \in \mathbb{Z}\}$  satisfies the Y-system  $\mathbb{Y}(X, X')$  associated to a pair of *ADE* Dynkin diagrams.

**Theorem 3.2.** [K] *Y-system is periodic and*

$$(3.3) \quad Y_{\mathbf{i}}(u + 2(h + h')) = Y_{\mathbf{i}}(u).$$

Let  $(y_{\mathbf{i}})_{\mathbf{i} \in \mathbf{I}}$  be indeterminates and set

$$(3.4) \quad Y_{\mathbf{i}}(0) = y_{\mathbf{i}}, \mathbf{i} \in \mathbf{I}_+$$

$$(3.5) \quad Y_{\mathbf{i}}(-1) = y_{\mathbf{i}}^{-1}, \mathbf{i} \in \mathbf{I}_-.$$

Then each  $Y_{\mathbf{i}}(u)$  with  $(\mathbf{i}, u) \in P_+$  can be regarded as a rational function in  $y_{\mathbf{i}}$ 's. Let  $\mathbb{Q}(y)$  be the field of rational functions in  $y_{\mathbf{i}}$ 's.

**Theorem 3.3.** [Nak] *Let  $(\mathbf{i}, u) \in P_+$ . Then*

$$(3.6) \quad Y_{\mathbf{i}}(u) = G_{\mathbf{i}}(u)T_{\mathbf{i}}(u) \in \mathbb{Q}(y)$$

where  $G_{\mathbf{i}}(u) \in \mathbb{Q}(y)$  satisfies  $G_{\mathbf{i}}(u)|_{(0, \dots, 0)} = 1$  and  $T_{\mathbf{i}}(u) \neq 1$  is a positive or negative monomial in  $y_{\mathbf{i}}$ 's, i.e.  $T_{\mathbf{i}}(u)$  can be written as a product of  $y_{\mathbf{i}}$ 's or as a product of  $y_{\mathbf{i}}^{-1}$ 's.

Finally, I state a property of Y-system which Nakanishi called the constancy condition in [Nak].

**Theorem 3.4.** [Nak, proposition 3.2 (i)]

Let  $S_+ = \{(\mathbf{i}, u) | 0 \leq u \leq 2(h + h') - 1, (\mathbf{i}, u) \in P_+\}$

$$(3.7) \quad \sum_{(\mathbf{i}, u) \in S_+} Y_{\mathbf{i}}(u) \wedge (1 + Y_{\mathbf{i}}(u)) = 0 \in \Lambda^2 \mathbb{Q}(y)^*.$$

**3.2. extending results to foldings of ADE Dynkin diagrams.** The results in the previous section can be extended to include all foldings of ADE diagrams almost trivially.

First note that for a pair  $(X, X')$  of directed graphs with the index sets  $I$  and  $I'$ , we can redefine the Y-system in terms of their adjacency matrices of graphs as follows :

$$(3.8) \quad Y_{ii'}(u-1)Y_{ii'}(u+1) = \frac{\prod_{j \in I} (1 + Y_{ji'}(u))^{\mathcal{I}(X)_{ij}}}{\prod_{j' \in I'} (1 + Y_{ij'}(u)^{-1})^{\mathcal{I}(X')_{i'j'}}}.$$

Let  $X$  be a Dynkin diagram of ADE type. We can regard it as a directed graph with the adjacency matrix  $\mathcal{I}(X)$ . For a group  $G$  of diagram automorphisms of  $X$ , we can define a quotient diagram  $\bar{X} = X/G$  as follows :  $\bar{X}$  has the vertex set  $\bar{I}$ , the orbit of  $I$  under  $G$ .  $(\bar{i}, \bar{j})$  is an edge of  $\bar{X}$  if  $(i, j)$  is an edge of  $G$  and the multiplicity  $\mathcal{I}(\bar{X})_{\bar{i}\bar{j}}$  is defined as the number of preimages of  $\bar{j}$  for a fixed representative  $i$  of  $\bar{i}$ . Let us call  $\bar{X}$  the folding of  $X$  by  $G$ .

Note that  $\bar{X}$  is generally a directed graph and its adjacency matrix  $\mathcal{I}(\bar{X})$  may not be symmetric. Let us call the matrix  $\mathcal{C}(\bar{X}) = 2\text{id} - \mathcal{I}(\bar{X})$  the Cartan matrix of  $\bar{X}$ . If  $G$  is a trivial group, we just get  $\bar{X} = X$ . The Coxeter number of  $\bar{X}$  is same as the Coxeter number of  $X$ .

The tadpole graph  $T_r$  is obtained as the folding of  $X = A_{2r}$  by the diagram automorphism group of order 2. The Cartan matrix  $\mathcal{C}(T_r)$  is the same as  $\mathcal{C}(A_r)$  except that a diagonal entry is 1 instead of 2 because  $T_r$  diagram has a loop.

$\bar{X}$  inherits the bipartite decomposition  $\bar{I} = \bar{I}_+ \cup \bar{I}_-$  of  $X$  except when  $\bar{X} = T_n$  because  $T_n$  diagram has a loop and cannot be bipartite, in which case, we just set  $\bar{I} = \bar{I}_+ = \bar{I}_-$ .

Let  $(\bar{X}, \bar{X}')$  be a pair of foldings of  $ADE$  Dynkin diagrams. Let  $\bar{\mathbf{I}} = \bar{I} \times \bar{I}'$  and define a decomposition  $\bar{\mathbf{I}} = \bar{\mathbf{I}}_+ \cup \bar{\mathbf{I}}_-$  where  $\bar{\mathbf{I}}_+ = (\bar{I}_+ \times \bar{I}'_+) \cup (\bar{I}_- \times \bar{I}'_-)$  and  $\bar{\mathbf{I}}_- = (\bar{I}_+ \times \bar{I}'_-) \cup (\bar{I}_- \times \bar{I}'_+)$ . When  $\bar{X} = T_r$  or  $\bar{X}' = T_{r'}$ , we just get  $\bar{\mathbf{I}} = \bar{\mathbf{I}}_+ = \bar{\mathbf{I}}_-$ .

When  $\bar{X} = T_r$  or  $\bar{X}' = T_{r'}$ , just set  $\bar{P}_+ = \bar{P}_- = \bar{\mathbf{I}} \times \mathbb{Z}$ . Otherwise, we can define  $\bar{P}_\pm$  similarly as in the previous section.

From a solution  $(Y_{\mathbf{i}}(u))_{(\mathbf{i}, u) \in \mathbf{I} \times \mathbb{Z}}$  of  $\mathbb{Y}(X, X')$ , one can obtain a solution  $(Y_{\bar{\mathbf{i}}}(u))_{(\bar{\mathbf{i}}, u) \in \bar{\mathbf{I}} \times \mathbb{Z}}$  of the Y-system  $\mathbb{Y}(\bar{X}, \bar{X}')$  by just setting  $Y_{\bar{\mathbf{i}}}(u) = Y_{\mathbf{i}}(u)$ .

Now we restate the theorems in the previous section for a Y-system  $\mathbb{Y}(\bar{X}, \bar{X}')$  associated to a pair of foldings of  $ADE$  Dynkin diagrams.

**Theorem 3.5.** *The theorems 3.2 and 3.3 hold true for a solution of  $\mathbb{Y}(\bar{X}, \bar{X}')$ .*

Let  $(y_{\bar{\mathbf{i}}})_{\bar{\mathbf{i}} \in \bar{\mathbf{I}}}$  be indeterminates and set

$$(3.9) \quad Y_{\bar{\mathbf{i}}}(0) = y_{\bar{\mathbf{i}}}, \bar{\mathbf{i}} \in \bar{\mathbf{I}}_+$$

$$(3.10) \quad Y_{\bar{\mathbf{i}}}(-1) = y_{\bar{\mathbf{i}}}^{-1}, \bar{\mathbf{i}} \in \bar{\mathbf{I}}_-.$$

Then again each  $Y_{\bar{\mathbf{i}}}(u)$  with  $(\bar{\mathbf{i}}, u) \in P_+$  can be regarded as an element of  $\mathbb{Q}(y)$ , the field of rational functions in  $y_{\bar{\mathbf{i}}}$ 's.

Let us define  $d_{\bar{\mathbf{i}}}(u)$  as the number of the preimages of  $(\bar{\mathbf{i}}, u)$  under the quotient map  $\mathbf{I} \times \mathbb{Z} \rightarrow \bar{\mathbf{I}} \times \mathbb{Z}$ . Then the theorem 3.4 can be restated as the following.

**Theorem 3.6.** *Let  $\bar{S}_+ = \{(\bar{\mathbf{i}}, u) | 0 \leq u \leq 2(h + h') - 1, (\bar{\mathbf{i}}, u) \in P_+\}$*

$$(3.11) \quad \sum_{(\bar{\mathbf{i}}, u) \in \bar{S}_+} d_{\bar{\mathbf{i}}}(u) (Y_{\bar{\mathbf{i}}}(u) \wedge (1 + Y_{\bar{\mathbf{i}}}(u))) = 0 \in \Lambda^2 \mathbb{Q}(y)^*$$

*In other words, the element*

$$(3.12) \quad \sum_{(\bar{\mathbf{i}}, u) \in \bar{S}_+} d_{\bar{\mathbf{i}}}(u) \left[ \frac{Y_{\bar{\mathbf{i}}}(u)}{1 + Y_{\bar{\mathbf{i}}}(u)} \right]$$

*of the group ring of  $\mathbb{Q}(y)$  is an element of the Bloch group  $\mathcal{B}(\mathbb{Q}(y))$ .*

Now I prove that Y-system produces a torsion element of the Bloch group.

**Proposition 3.7.** *Let  $f_{\bar{\mathbf{i}}}(u) = \frac{Y_{\bar{\mathbf{i}}}(u)}{1 + Y_{\bar{\mathbf{i}}}(u)} \in \mathbb{Q}(y)$ . Then*

$$(3.13) \quad \sum_{(\bar{\mathbf{i}}, u) \in \bar{S}_+} d_{\bar{\mathbf{i}}}(u) D(f_{\bar{\mathbf{i}}}(u)|_{\mathbf{x}}) = 0$$

*for any  $\mathbf{x} = (x_{\bar{\mathbf{i}}}) \in \mathbb{C}^n$  where  $n = rr'$ .*

*Proof.* To employ the proposition 2.2, we check the following condition

$$(3.14) \quad \sum_{(\bar{i}, u) \in \bar{S}_+} d_{\bar{i}}(u)(f_{\bar{i}}(u) \wedge (1 - f_{\bar{i}}(u))) = 0$$

This is equivalent to

$$\sum_{(\bar{i}, u) \in \bar{S}_+} d_{\bar{i}}(u) \left( \left( \frac{Y_{\bar{i}}(u)}{1 + Y_{\bar{i}}(u)} \right) \wedge \left( \frac{1}{1 + Y_{\bar{i}}(u)} \right) \right) = 0$$

which reduces to the constancy condition of Y-system

$$\sum_{(\bar{i}, u) \in \bar{S}_+} d_{\bar{i}}(u)(Y_{\bar{i}}(u) \wedge (1 + Y_{\bar{i}}(u))) = 0.$$

Thus the condition 3.14 is satisfied by the theorem 3.6.

Now all we have to check is that there is a point  $\mathbf{a} \in \mathbb{C}^n$  such that  $\sum_{(\bar{i}, u) \in \bar{S}_+} D(f_{\bar{i}}(u)|_{\mathbf{a}}) = 0$ . By the theorem 3.5,  $f_{\bar{i}}(u) = \frac{G_{\bar{i}}(u)T_{\bar{i}}(u)}{1 + G_{\bar{i}}(u)T_{\bar{i}}(u)}$ . Since  $G_{\bar{i}}(u)|_{(0, \dots, 0)} = 1$  and  $T_{\bar{i}}(u) \neq 1$  is a positive or negative monomial in  $y_{\bar{i}}$ 's,  $f_{\bar{i}}(u)|_{(0, \dots, 0)}$  is always 0 or 1 depending on whether  $T_{\bar{i}}(u)$  is positive or negative. So we can simply choose  $\mathbf{a} = (0, \dots, 0)$  and then  $\sum_{(\bar{i}, u) \in \bar{S}_+} D(f_{\bar{i}}(u)|_{\mathbf{a}}) = 0$  and therefore  $\sum_{(\bar{i}, u) \in \bar{S}_+} D(f_{\bar{i}}(u)|_{\mathbf{x}}) = 0$  for any  $\mathbf{x} = (x_{\bar{i}}) \in \mathbb{C}^n$  by the proposition 2.2.  $\square$

*Remark 3.8.* This proposition generalizes [FS2, theorem 2] and [FG, corollary 6.14.]. If we count how many of  $f_{\bar{i}}(u)|_{(0, \dots, 0)}$  becomes 1 in the above argument, we can obtain a proof of the dilogarithm identities for central charges of certain conformal field theories. See [Nak].

**Corollary 3.9.** *Let  $(\bar{X}, \bar{X}')$  be a pair of ADET diagrams. If  $(y_{\bar{i}}) \in F^n$  is a solution of the constant Y-system  $\mathbb{Y}(\bar{X}, \bar{X}')$  for a number field  $F$ ,*

$$(3.15) \quad \sum_{\bar{i} \in \bar{\mathbf{I}}} \left[ \frac{y_{\bar{i}}}{1 + y_{\bar{i}}} \right] \in \mathcal{B}(F)$$

*is a torsion element of the Bloch group  $\mathcal{B}(F)$ .*

*Proof.* Let  $\sigma : F \hookrightarrow \mathbb{C}$  be an embedding. By the proposition 3.7, we know

$$(3.16) \quad \sum_{(\bar{i}, u) \in \bar{S}_+} d_{\bar{i}}(u) D\left(\sigma\left(\frac{y_{\bar{i}}}{1 + y_{\bar{i}}}\right)\right) = 0$$

Note that when  $(\bar{X}, \bar{X}')$  is given by a pair of ADET diagrams,  $d_{\bar{i}}(u)$  is same for all  $(\bar{i}, u)$ . If  $(\bar{X}, \bar{X}') = (T_r, T_{r'})$ ,  $d_{\bar{i}}(u) = 2$  and  $d_{\bar{i}}(u) = 1$  otherwise.

Thus

$$(3.17) \quad \sum_{\bar{i} \in \bar{\mathbf{I}}} D\left(\sigma\left(\frac{y_{\bar{i}}}{1 + y_{\bar{i}}}\right)\right) = 0.$$



Since this is true for any  $\sigma : F \hookrightarrow \mathbb{C}$ ,  $\sum_{\mathbf{i} \in \bar{\mathbf{I}}} [\frac{y_{\mathbf{i}}}{1+y_{\mathbf{i}}}]$  is a torsion element of the Bloch group.  $\square$

#### 4. PROOF OF THE MAIN THEOREM

We can now prove the theorem 1.3. Let  $(X, X')$  be a pair of *ADET* Dynkin diagrams. First I relate a solution of the equation  $\mathbf{x} = (1 - \mathbf{x})^A$  to the Y-system  $\mathbb{Y}(X, X')$ .

**Proposition 4.1.** *Let  $A = \mathcal{C}(X) \otimes \mathcal{C}(X')^{-1}$ . If  $x = (x_{\mathbf{i}})$  is a solution to  $\mathbf{x} = (1 - \mathbf{x})^A$  in a number field  $F$ ,  $y_{\mathbf{i}} = \frac{x_{\mathbf{i}}}{1-x_{\mathbf{i}}}$  is a solution to the constant Y-system  $\mathbb{Y}(X, X')$ .*

*Proof.* Let us rewrite the equation  $\mathbf{x} = (1 - \mathbf{x})^A$ .

$$(4.1) \quad x_{\mathbf{i}} = \prod_{\mathbf{j} \in \bar{\mathbf{I}}} (1 - x_{\mathbf{j}})^{(\mathcal{C}(X) \otimes \mathcal{C}(X')^{-1})_{\mathbf{ij}}}$$

$$(4.2) \quad x_{ii'} = \prod_{(j, j') \in I \times I'} (1 - x_{jj'})^{(\mathcal{C}(X) \otimes \mathcal{C}(X')^{-1})_{ij}}$$

$$(4.3) \quad \prod_{j' \in I'} x_{ij'}^{\mathcal{C}(X')_{i'j'}} = \prod_{j \in I} (1 - x_{ji'})^{\mathcal{C}(X)_{ij}}$$

Since  $A$  is a positive definite matrix, all diagonal entries are positive. So from the equation (4.1) one can see that  $x_{\mathbf{i}}$  is neither 0 nor 1. Now use the change of variables  $y_{\mathbf{i}} = \frac{x_{\mathbf{i}}}{1-x_{\mathbf{i}}}$  or  $x_{\mathbf{i}} = \frac{y_{\mathbf{i}}}{1+y_{\mathbf{i}}} = \frac{1}{1+y_{\mathbf{i}}^{-1}}$ .

$$(4.4) \quad \prod_{j' \in I'} \left( \frac{1}{1 + y_{ij'}^{-1}} \right)^{\mathcal{C}(X')_{i'j'}} = \prod_{j \in I} \left( \frac{1}{1 + y_{ji'}} \right)^{\mathcal{C}(X)_{ij}}$$

$$(4.5) \quad 1 = \frac{\prod_{j \in I} (1 + y_{ji'})^{-\mathcal{C}(X)_{ij}}}{\prod_{j' \in I'} (1 + y_{ij'}^{-1})^{-\mathcal{C}(X')_{i'j'}}}$$

This can be written as

$$(4.6) \quad \left( \frac{1}{1 + y_{ii'}^{-1}} \right)^2 (1 + y_{ii'})^2 = \frac{\prod_{j \in I} (1 + y_{ji'})^{\mathcal{I}(X)_{ij}}}{\prod_{j' \in I'} (1 + y_{ij'}^{-1})^{\mathcal{I}(X')_{i'j'}}}.$$

Thus we finally get a constant Y-system

$$(4.7) \quad y_{ii'}^2 = \frac{\prod_{j \in I} (1 + y_{ji'})^{\mathcal{I}(X)_{ij}}}{\prod_{j' \in I'} (1 + y_{ij'}^{-1})^{\mathcal{I}(X')_{i'j'}}}.$$

□

Now we can finish the proof of the theorem 1.3.

*Proof.* Let  $x = (x_i)$  be a solution to  $\mathbf{x} = (1 - \mathbf{x})^A$ . By proposition 4.1,  $y_i = \frac{x_i}{1-x_i}$  is a solution to the constant Y-system  $\mathbb{Y}(X, X')$ . The theorem follows from the corollary 3.9. □

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